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Certain New Ramanujan Type Theta Function Identities

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Abstract

In this article, we prove an interesting summation formula. From this summation formula we prove certain theta function identities which are analogous to that of Ramanujan.

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1 Introduction

For any complex numbers a and q, let

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),\ |q|<1$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},\ \text{n is any integer.}$$

In Chapter 16 of his second notebook, Ramanujan defined

$$f(a,b) = \sum_{n=-\infty}^{\infty}(-1)^n a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |ab|<1.$$

By the well known Jacobi triple product identity, we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.1}$$

Further, Ramanujan also defines

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = (q;q)_\infty.$$

In scattered places of his notebook, Ramanujan recorded many identities of the following type :

$$\varphi^2(q)\,\varphi^2(q^3) = 1 + 4\left(\frac{q}{1-q} + \frac{4q^4}{1-q^4} + \frac{5q^5}{1-q^5} + \frac{7q^7}{1-q^7} + \dots\right) \tag{1.2}$$

and

$$\frac{f^5(-q)}{f(-q^5)} = 1 - 5\left(\frac{q}{1+q} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1+q^4} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \frac{11q^{11}}{1+q^{11}} - \frac{12q^{12}}{1+q^{12}} - \dots\right). \tag{1.3}$$

Ramanujan recorded (1.2) and (1.3) respectively in [4, p. 225] and [4, p. 229].

In this paper, we derive the following summation formula :

$$\sum_{n=-\infty}^{\infty} \frac{a^{-n}q^n}{(1-aq^n)^2} = \frac{(q;q)_\infty^4}{(a;q)_\infty^2\left(\frac{q}{a};q\right)_\infty^2}, \quad |q| < |a| < 1. \tag{1.4}$$

From this we deduce following identities analogous to (1.2) and (1.3).

$$\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{5n}}{(1-q^{25n+20})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{20n+20}}{(1-q^{25n+30})}\right) - q^2\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{10n}}{(1-q^{25n+15})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{15n+15}}{(1-q^{25n+30})}\right) = \frac{f_1 f_{25}^4}{f_5} + q\frac{f_{25}^5}{f_5}, \tag{1.5}$$

$$4q^8\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{5n}}{(1-q^{50n+45})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{45n+45}}{(1-q^{50n+55})}\right) + 4q^2\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{5n}}{(1-q^{50n+35})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{35n+35}}{(1-q^{50n+65})}\right) = \frac{f_1^4}{f_2^2} + \frac{f_{25}^4}{f_{50}^2} - 2\frac{f_1^2}{f_2}\frac{f_{25}^2}{f_{50}} - 4q^5\frac{f_5 f_{50}^3}{f_{10} f_{25}}, \tag{1.6}$$

$$\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{12n+10})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{10n+10}}{(1-q^{12n+14})}\right) + q\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{12n+8})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{8n+8}}{(1-q^{12n+16})}\right) = \frac{f_4^6 f_6^2 f_{12}^2}{f_2^4 f_8^2} + q\frac{f_{12}^6}{f_2^2 f_8^2}, \tag{1.7}$$

$$\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{12n+10})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{10n+10}}{(1-q^{12n+14})}\right) - q\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{12n+8})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{8n+8}}{(1-q^{12n+16})}\right) = \frac{f_4^6 f_6^2 f_{12}^2}{f_2^4 f_8^2} - q\frac{f_{12}^6}{f_2^2 f_8^2}, \tag{1.8}$$

$$q\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{8n+7})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{7n+7}}{(1-q^{8n+9})}\right) - \left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{8n+5})} + \sum_{n=0}^{\infty}\frac{(n+1)\,q^{5n+5}}{(1-q^{8n+11})}\right) = \frac{f_2 f_4^4 f_8^2}{f_1^3}, \tag{1.9}$$

$$q\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{8n+7})}+\sum_{n=0}^{\infty}\frac{(n+1)\,q^{7n+7}}{(1-q^{8n+9})}\right)+\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{8n+5})}+\sum_{n=0}^{\infty}\frac{(n+1)\,q^{5n+5}}{(1-q^{8n+11})}\right)=\frac{f_2^8f_8^2}{f_1^5f_4^3},\quad (1.10)$$

and

$$q^2\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{12n+11})}+\sum_{n=0}^{\infty}\frac{(n+1)\,q^{11n+11}}{(1-q^{12n+13})}\right)+\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{n}}{(1-q^{12n+7})}+\sum_{n=0}^{\infty}\frac{(n+1)\,q^{7n+7}}{(1-q^{12n+17})}\right)=\frac{f_2^3f_6^3}{f_1^2}.\quad (1.11)$$

We prove (1.4) by using the following well known Ramanujan's ${}_1\psi_1$ summation formula :

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(az;q)_\infty\left(\frac{q}{az};q\right)_\infty\left(\frac{b}{a};q\right)_\infty(q;q)_\infty}{(z;q)_\infty\left(\frac{b}{az};q\right)_\infty\left(\frac{q}{a};q\right)_\infty(b;q)_\infty},\quad |q|<1,\ \left|\frac{b}{a}\right|<|z|<1.\quad (1.12)$$

For a quite different proof of (1.4) one can refer [6]. In Section 2, we recall the known identities which are required to prove our main result. In Section 3, we prove (1.4) and establish the seven new identities (1.5) - (1.11).

2 Preliminary Results

For convenience, we set $f(-q^n)=f_n$ for positive integer n. It is easy to see that

$$\varphi(q)=\frac{f_2^5}{f_1^2f_4^2},\quad \varphi(-q)=\frac{f_1^2}{f_2},\quad \psi(q)=\frac{f_2^2}{f_1},\quad f(q)=\frac{f_2^3}{f_1f_4},\quad \text{and}\quad \chi(q)=\frac{f_2^2}{f_1f_4}.\quad (2.1)$$

We require the following theta function identities :

$$\varphi\left(q^{\frac{1}{5}}\right)-\varphi\left(q^5\right)=2q^{\frac{1}{5}}f\left(q^3,q^7\right)+2q^{\frac{4}{5}}f\left(q,q^9\right),\quad (2.2)$$

$$f\left(-q\right)\left\{f\left(-q^{\frac{1}{5}}\right)+q^{\frac{1}{5}}f\left(-q^5\right)\right\}=f^2\left(-q^2,-q^3\right)-q^{\frac{2}{5}}f^2\left(-q,-q^4\right),\quad (2.3)$$

$$f\left(-q^5,-q^7\right)-qf\left(-q,-q^{11}\right)=\frac{\varphi\left(q^3\right)}{\chi\left(q\right)},\quad (2.4)$$

$$f\left(-q^5,-q^7\right)+qf\left(-q,-q^{11}\right)=f\left(q\right),\quad (2.5)$$

and

$$T^2\left(q^2\right)+qS^2\left(q^2\right)=\frac{1}{f_8^2}\left[\psi^2\left(q^2\right)+q\psi^2\left(q^6\right)\right],\quad (2.6)$$

$$S_1\left(q\right)S_1\left(q\right)+qT_1\left(q\right)T_1\left(q\right)=\left\{\frac{f_2f_2}{f_1f_4}\right\}^3,\quad (2.7)$$

$$S_1\left(q\right)S_1\left(q\right)-qT_1\left(q\right)T_1\left(q\right)=\frac{f_4f_4}{f_1f_2}\left\{\frac{f_4}{f_8}\right\}^2,\quad (2.8)$$

where

$$S(q)=\frac{\left(-q;q^2\right)_\infty\left(q^6;q^6\right)_\infty\left(q;q^6\right)_\infty\left(q^5;q^6\right)_\infty}{\left(q^2;q^2\right)_\infty},$$

$$T(q) = \frac{\left(-q;q^2\right)_\infty \left(q^6;q^6\right)_\infty \left(q^2;q^6\right)_\infty \left(q^4;q^6\right)_\infty}{\left(q^2;q^2\right)_\infty},$$

$$S_1(q) = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty},$$

and

$$T_1(q) = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}.$$

For proofs of (2.2) and (2.3), see [2, p. 262]. For proofs of (2.4) and (2.5), one can refer [5]. (2.6) is found in [1]. For proofs of (2.7) and (2.8), one may refer [3].

3 Main results

Proof of (1.4) **:** Setting $b = aq$ in (1.12), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{(az)_\infty \left(\frac{q}{az}\right)_\infty (q)_\infty^2}{(z)_\infty \left(\frac{q}{z}\right)_\infty (a)_\infty \left(\frac{q}{a}\right)_\infty}.$$

Differentiating the above, w.r.t a, and using the fact that $f^{'}(a) = f(a)\frac{d}{da}\log f(a)$ in the right-hand side of the above, we find that

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^n}{\left(1-aq^n\right)^2}$$
$$= \frac{(azq)_\infty \left(\frac{q}{az}\right)_\infty (q)_\infty^2}{(z)_\infty \left(\frac{q}{z}\right)_\infty (a)_\infty \left(\frac{q}{a}\right)_\infty} \left[-z + (1-az)\left\{-\sum_{n=1}^{\infty} \frac{zq^n}{1-azq^n} + \sum_{n=0}^{\infty} \frac{\frac{q^{n+1}}{a^2 z}}{1-\frac{q^{n+1}}{az}} + \sum_{n=0}^{\infty} \frac{q^n}{1-aq^n} - \sum_{n=0}^{\infty} \frac{\frac{q^{n+1}}{a^2}}{1-\frac{q^{n+1}}{a}}\right\}\right].$$

Setting $z = \frac{1}{a}$ in the above, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{\left(\frac{q}{a}\right)^n}{\left(1-aq^n\right)^2} = \frac{(q)_\infty^4}{(a)_\infty^2 \left(\frac{q}{a}\right)_\infty^2}.$$

This completes the proof of (1.4).

Proof of (1.5) **:** Using (1.1) on the right-hand side of the (1.4), we find that

$$\sum_{n=-\infty}^{\infty} \frac{\left(\frac{q}{a}\right)^n}{\left(1-aq^n\right)^2} = \frac{(q;q)_\infty^6}{f^2\left(-a, \frac{-q}{a}\right)}. \tag{3.1}$$

Replacing q by q^5 in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}a^{-n}}{(1-aq^{5n})^2} = \frac{f_5^6}{f^2\left(-a, \frac{-q^5}{a}\right)}. \tag{3.2}$$

Setting $a = q$ in (3.2), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{(1-q^{5n+1})^2} = \frac{f_5^6}{f^2(-q,-q^4)}. \tag{3.3}$$

Setting $a = q^2$ in (3.2), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{5n+2})^2} = \frac{f_5^6}{f^2(-q^2,-q^3)}. \tag{3.4}$$

Subtracting $q^{\frac{2}{5}}$ times of (3.4) from (3.3), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{(1-q^{5n+1})^2} - q^{\frac{2}{5}} \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{5n+2})^2} = f_5^6 \left[\frac{f^2(-q^2,-q^3) - q^{\frac{2}{5}} f^2(-q,-q^4)}{f^2(-q,-q^4)\, f^2(-q^2,-q^3)}\right].$$

Using (2.3) in the right-hand side of the above, we find that

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{(1-q^{5n+1})^2} - q^{\frac{2}{5}} \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{5n+2})^2} = \frac{f_5^6 f_1 \left[f\left(-q^{\frac{1}{5}}\right) + q^{\frac{1}{5}} f_5\right]}{f_1^2 f_5^2}.$$

Changing q to q^5 in the above and simplifying right-hand side, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{20n}}{(1-q^{25n+5})^2} - q^2 \sum_{n=-\infty}^{\infty} \frac{q^{15n}}{(1-q^{25n+10})^2} = \frac{f_1 f_{25}^4}{f_5} + q\frac{f_{25}^5}{f_5}.$$

Observing that $\frac{1}{(1-q)^2} = \frac{d}{dq}\left(\sum\limits_{n=0}^{\infty} q^n\right) = \sum\limits_{n=1}^{\infty} nq^{n-1}$, we obtain

$$\left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^{5n}}{(1-q^{25n+20})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{20n+20}}{(1-q^{25n+30})}\right) - q^2 \left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^{10n}}{(1-q^{25n+15})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{15n+15}}{(1-q^{25n+35})}\right)$$
$$= \frac{f_1 f_{25}^4}{f_5} + q\frac{f_{25}^5}{f_5}.$$

This completes the proof of (1.5).

Proof of (1.6) **:** Replacing q by q^{10} in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n}a^{-n}}{(1-aq^{10n})^2} = \frac{f_{10}^6}{f^2\left(-a, \frac{-q^{10}}{a}\right)}. \tag{3.5}$$

Setting $a = q$ in (3.5), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{9n}}{(1-q^{10n+1})^2} = \frac{f_{10}^6}{f^2(-q,-q^9)}. \tag{3.6}$$

Setting $a=q^3$ in (3.5), we obtain

$$\sum_{n=-\infty}^{\infty}\frac{q^{7n}}{(1-q^{10n+3})^2}=\frac{f_{10}^6}{f^2(-q^3,-q^7)}. \tag{3.7}$$

Adding $q^{\frac{6}{5}}$ times of (3.6) and (3.7), we obtain

$$q^{\frac{6}{5}}\sum_{n=-\infty}^{\infty}\frac{q^{9n}}{(1-q^{10n+1})^2}+\sum_{n=-\infty}^{\infty}\frac{q^{7n}}{(1-q^{10n+3})^2}=\frac{f_{10}^6\left[q^{\frac{6}{5}}f^2(-q,-q^9)+f^2(-q^3,-q^7)\right]}{f^2(-q,-q^9)\,f^2(-q^3,-q^7)},$$

Employing (2.2) on the right-hand side of the above, we obtain

$$4q^{\frac{8}{5}}\sum_{n=-\infty}^{\infty}\frac{q^{9n}}{(1-q^{10n+1})^2}+4q^{\frac{2}{5}}\sum_{n=-\infty}^{\infty}\frac{q^{7n}}{(1-q^{10n+3})^2}$$
$$=\frac{f_2^2f_5^2}{f_1^2}\left[\varphi\left(-q^{\frac{1}{5}}\right)+\varphi^2(-q^5)-2\varphi\left(-q^{\frac{1}{5}}\right)\varphi(-q^5)-4q\frac{f_1f_{10}^3}{f_2f_5}\right].$$

Changing q by q^5 in the above and using (2.1) on the right-hand side of the above, we obtain

$$4q^8\sum_{n=-\infty}^{\infty}\frac{q^{9n}}{(1-q^{10n+1})^2}+4q^2\sum_{n=-\infty}^{\infty}\frac{q^{7n}}{(1-q^{10n+3})^2}=\frac{f_1^4}{f_2^2}+\frac{f_{25}^4}{f_{50}^2}-2\frac{f_1^2}{f_2}\frac{f_{25}^2}{f_{50}}-4q^5\frac{f_5f_{50}^3}{f_{10}f_{25}},$$

which is equivalent to

$$4q^8\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{5n}}{(1-q^{50n+45})}+\sum_{n=0}^{\infty}\frac{(n+1)\,q^{45n+45}}{(1-q^{50n+55})}\right)+4q^2\left(\sum_{n=0}^{\infty}\frac{(n+1)\,q^{5n}}{(1-q^{50n+35})}+\sum_{n=0}^{\infty}\frac{(n+1)\,q^{35n+35}}{(1-q^{50n+65})}\right)$$
$$=\frac{f_1^4}{f_2^2}+\frac{f_{25}^4}{f_{50}^2}-2\frac{f_1^2}{f_2}\frac{f_{25}^2}{f_{50}}-4q^5\frac{f_5f_{50}^3}{f_{10}f_{25}}.$$

This completes the proof of (1.6).

Proof of (1.7)**:** Replacing q by q^6 in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty}\frac{q^{6n}a^{-n}}{(1-aq^{6n})^2}=\frac{f_6^6}{f^2\left(-a,\frac{-q^6}{a}\right)}. \tag{3.8}$$

Setting $a=q$ in (3.8), we obtain

$$\sum_{n=-\infty}^{\infty}\frac{q^{5n}}{(1-q^{6n+1})^2}=\frac{f_6^6}{f^2(-q,-q^5)}. \tag{3.9}$$

Setting $a=q^2$ in (3.8), we obtain

$$\sum_{n=-\infty}^{\infty}\frac{q^{4n}}{(1-q^{6n+2})^2}=\frac{f_6^6}{f^2(-q^2,-q^4)}. \tag{3.10}$$

Adding (3.9) and $q^{\frac{1}{2}}$ times of (3.10), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{\left(1-q^{6n+1}\right)^2} + q^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{\left(1-q^{6n+2}\right)^2} = f_6^6 \left[\frac{f^2\left(-q^2,-q^4\right) + q^{\frac{1}{2}} f^2\left(-q,-q^5\right)}{f^2\left(-q,-q^5\right) f^2\left(-q^2,-q^4\right)} \right].$$

Changing q to $q^{\frac{1}{2}}$ in (2.6) and using the resulting identity on the right-hand side of the above, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{\left(1-q^{6n+1}\right)^2} + q^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{\left(1-q^{6n+2}\right)^2} = \frac{f_6^4 f_3}{f_4^2 f_1} \left[\psi^2\left(q\right) + q^{\frac{1}{2}} \psi^2\left(q^3\right) \right].$$

Changing q to q^2 in the above and using (2.1) on the right-hand side of the above, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n}}{\left(1-q^{12n+2}\right)^2} + q \sum_{n=-\infty}^{\infty} \frac{q^{8n}}{\left(1-q^{12n+4}\right)^2} = \frac{f_4^6 f_6^2 f_{12}^2}{f_2^4 f_8^2} + q \frac{f_{12}^6}{f_2^2 f_8^2},$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{\left(1-q^{12n+10}\right)} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{10n+10}}{\left(1-q^{12n+14}\right)} \right) + q \left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{\left(1-q^{12n+8}\right)} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{8n+8}}{\left(1-q^{12n+16}\right)} \right)$$
$$= \frac{f_4^6 f_6^2 f_{12}^2}{f_2^4 f_8^2} + q \frac{f_{12}^6}{f_2^2 f_8^2}.$$

This completes the proof of (1.7).

Proof of (1.8) **:** Replacing q by q^6 in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{6n} a^{-n}}{\left(1-aq^{6n}\right)^2} = \frac{f_6^6}{f^2\left(-a, \frac{-q^6}{a}\right)}. \tag{3.11}$$

Setting $a = q$ in (3.11), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{\left(1-q^{6n+1}\right)^2} = \frac{f_6^6}{f^2\left(-q,-q^5\right)}. \tag{3.12}$$

Setting $a = q^2$ in (3.11), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{\left(1-q^{6n+2}\right)^2} = \frac{f_6^6}{f^2\left(-q^2,-q^4\right)}. \tag{3.13}$$

Subtracting $q^{\frac{1}{2}}$ times of (3.13) from (3.12), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{\left(1-q^{6n+1}\right)^2} - q^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{\left(1-q^{6n+2}\right)^2} = f_6^6 \left[\frac{f^2\left(-q^2,-q^4\right) - q^{\frac{1}{2}} f^2\left(-q,-q^5\right)}{f^2\left(-q,-q^5\right) f^2\left(-q^2,-q^4\right)} \right].$$

Changing q to $q^{\frac{1}{2}}$ in (2.6) and using the resulting identity on the right-hand side of the above, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{\left(1-q^{6n+1}\right)^2} - q^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{\left(1-q^{6n+2}\right)^2} = \frac{f_6^6 f_3}{f_4^2 f_1 f_6} \left[\psi^2\left(q\right) - q^{\frac{1}{2}} \psi^2\left(q^3\right) \right].$$

Changing q to q^2 in the above and using (2.1) on the right-hand side of the above, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n}}{(1-q^{12n+2})^2} - q \sum_{n=-\infty}^{\infty} \frac{q^{8n}}{(1-q^{12n+4})^2} = \frac{f_4^6 f_6^2 f_{12}^2}{f_2^4 f_8^2} - q \frac{f_{12}^6}{f_2^2 f_8^2},$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{12n+10})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{10n+10}}{(1-q^{12n+14})}\right) - q \left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{12n+8})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{8n+8}}{(1-q^{12n+16})}\right)$$
$$= \frac{f_4^6 f_6^2 f_{12}^2}{f_2^4 f_8^2} - q \frac{f_{12}^6}{f_2^2 f_8^2}.$$

This completes the proof of (1.8).

Proof of (1.9) **:** Replacing q by q^8 in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{8n} a^{-n}}{(1-aq^{8n})^2} = \frac{f_8^6}{f^2\left(-a, \frac{-q^8}{a}\right)}. \tag{3.14}$$

Setting $a = q$ in (3.14), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{8n+1})^2} = \frac{f_8^6}{f^2(-q, -q^7)}. \tag{3.15}$$

Setting $a = q^3$ in (3.14), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{8n+3})^2} = \frac{f_8^6}{f^2(-q^3, -q^5)}. \tag{3.16}$$

Subtracting (3.16) from q times of (3.15), we obtain

$$q \sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{8n+1})^2} - \sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{8n+3})^2} = f_8^6 \left[\frac{f^2(-q^3, -q^5) - q f^2(-q, -q^7)}{f^2(-q, -q^7)\, f^2(-q^3, -q^5)}\right],$$

Employing (2.8) and (2.1) on the right-hand side of the above, we obtain

$$q \sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{8n+1})^2} - \sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{8n+3})^2} = \frac{f_2 f_4^4 f_8^2}{f_1^3},$$

which is equivalent to

$$q\left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{8n+7})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{7n+7}}{(1-q^{8n+9})}\right) - \left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{8n+5})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{5n+5}}{(1-q^{8n+11})}\right) = \frac{f_2 f_4^4 f_8^2}{f_1^3}.$$

This completes the proof of (1.9).

Proof of (1.10) **:** Replacing q by q^8 in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{8n}a^{-n}}{(1-aq^{8n})^2} = \frac{f_8^6}{f^2\left(-a, \frac{-q^8}{a}\right)}. \tag{3.17}$$

Setting $a = q$ in (3.17), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{8n+1})^2} = \frac{f_8^6}{f^2(-q, -q^7)}. \tag{3.18}$$

Setting $a = q^3$ in (3.17), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{8n+3})^2} = \frac{f_8^6}{f^2(-q^3, -q^5)}. \tag{3.19}$$

Adding q times of (3.18) and (3.19), we obtain

$$q\sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{8n+1})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{8n+3})^2} = f_8^6\left[\frac{f^2(-q^3, -q^5) + qf^2(-q, -q^7)}{f^2(-q, -q^7)\, f^2(-q^3, -q^5)}\right].$$

Employing (2.7) and (2.1) on the right-hand side of the above, we obtain

$$q\sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{8n+1})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{8n+3})^2} = \frac{f_2^8 f_8^2}{f_1^5 f_4^3},$$

which is equivalent to

$$q\left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{8n+7})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{7n+7}}{(1-q^{8n+9})}\right) + \left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{8n+5})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{5n+5}}{(1-q^{8n+11})}\right) = \frac{f_2^8 f_8^2}{f_1^5 f_4^3}.$$

This completes the proof of (1.10).

Proof of (1.11)**:** Replacing q by q^{12} in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{12n}a^{-n}}{(1-aq^{12n})^2} = \frac{f_{12}^6}{f^2\left(-a, \frac{-q^{12}}{a}\right)}. \tag{3.20}$$

Setting $a = q$ in (3.20), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{11n}}{(1-q^{12n+1})^2} = \frac{f_{12}^6}{f^2(-q, -q^{11})}. \tag{3.21}$$

Setting $a = q^5$ in (3.20), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{12n+5})^2} = \frac{f_{12}^6}{f^2(-q^5, -q^7)}. \tag{3.22}$$

Subtracting (3.22) from q^2 times of (3.21), we obtain

$$q^2 \sum_{n=-\infty}^{\infty} \frac{q^{11n}}{(1-q^{12n+1})^2} - \sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{12n+5})^2} = f_{12}^6 \left[\frac{f^2\left(-q^5,-q^7\right) - q^2 f^2\left(-q,-q^{11}\right)}{f^2\left(-q,-q^{11}\right) f^2\left(-q^5,-q^7\right)} \right].$$

Employing (2.4), (2.5) and (2.1) on the right-hand side of the above, we obtain

$$q^2 \sum_{n=-\infty}^{\infty} \frac{q^{11n}}{(1-q^{12n+1})^2} - \sum_{n=-\infty}^{\infty} \frac{q^{7n}}{(1-q^{12n+5})^2} = \frac{f_2^3 f_6^3}{f_1^2},$$

which is equivalent to

$$q^2 \left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{12n+11})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{11n+11}}{(1-q^{12n+13})} \right) + \left(\sum_{n=0}^{\infty} \frac{(n+1)\, q^n}{(1-q^{12n+7})} + \sum_{n=0}^{\infty} \frac{(n+1)\, q^{7n+7}}{(1-q^{12n+17})} \right) = \frac{f_2^3 f_6^3}{f_1^2}.$$

This complete the proof of (1.11).

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